

## Lesson 12. Derivatives and Integrals of Vector Functions

### 0 Warm up

**Example 1.** Find these derivatives and integrals.

a.  $\frac{d}{dt}(1 + t^3) =$

d.  $\int 2t \, dt =$

b.  $\frac{d}{dt}(\cos 2t) =$

e.  $\int 2 \cos t \, dt =$

c.  $\frac{d}{dt}(te^{-t}) =$

f.  $\int_0^{2\pi} 3 \, dt =$

### 1 In this lesson...

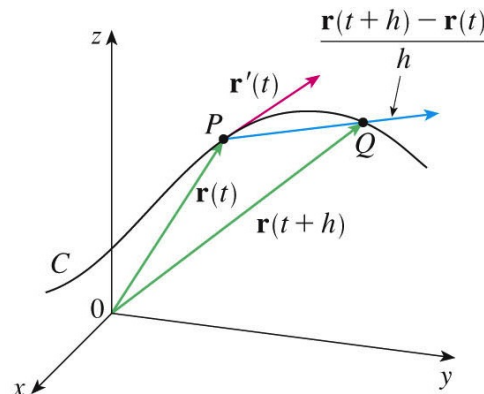
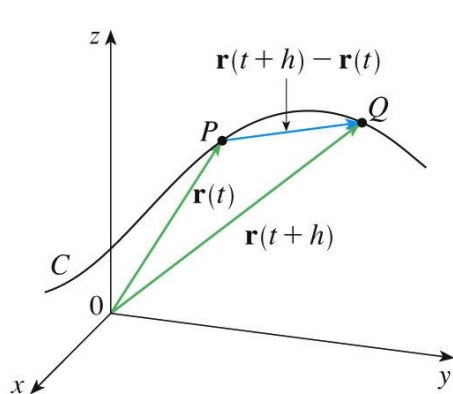
- What are the derivatives and integrals of vector functions?
- How can find the length of an arc?

### 2 Derivatives

- The **derivative** of  $\vec{r}$  is

$$\frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

- Note: the derivative of a vector function is also a vector



- Let  $C$  be the curve defined by  $\vec{r}$
- Let  $\vec{r}(t)$  be the position vector of  $P$
- The derivative  $\vec{r}'(t)$  is the direction vector of the line tangent to  $C$  at  $P$   
 $\Rightarrow$  Sometimes we refer to  $\vec{r}'(t)$  as the **tangent vector**

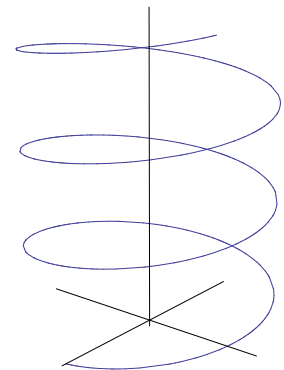
- The **unit tangent vector** is

- If  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ , where  $f$ ,  $g$ , and  $h$  are differentiable functions, then

**Example 2.**

- Find the derivative of  $\vec{r}(t) = \langle \cos 2t, 1 + t^3, te^{-t} \rangle$ .
- Find the unit tangent vector at the point where  $t = 0$ .

**Example 3.** Find parametric equations for the tangent line to the curve given by  $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$  at point  $(0, 1, \pi/2)$ .



- Differentiation rules:

$$\frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \vec{u}'(t) + \vec{v}'(t)$$

$$\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

$$\frac{d}{dt}(c\vec{u}(t)) = c\vec{u}'(t)$$

$$\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$$

$$\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$$

$$\frac{d}{dt}(\vec{u}(f(t))) = f'(t)\vec{u}'(f(t))$$

### 3 Integration

- Let  $f$ ,  $g$ , and  $h$  be continuous functions
- The **indefinite integral** of a vector function  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$  is

- The **definite integral** of a vector function  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$  from  $a$  to  $b$  is

- Note: the integral of a vector function is also a vector

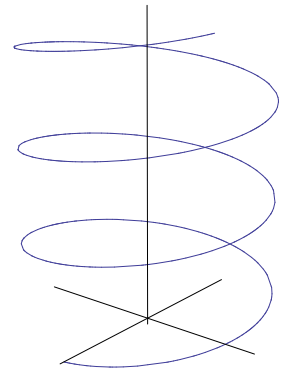
**Example 4.** Let  $\vec{r}(t) = \langle 2 \sin t, 2 \cos t, 2t \rangle$ . Find  $\int_0^{\pi/2} \vec{r}(t) dt$ .

#### 4 Arc length

- Let  $C$  be a curve with vector equation  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ ,  $a \leq t \leq b$
- What is the length of  $C$ ?
- The **arc length** of  $C$  is

- Similar for curves in 2D

**Example 5.** Let  $C$  be the curve defined by  $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ ,  $0 \leq t \leq 2\pi$ . Find the length of  $C$ .



**Example 6.** Let  $C$  be the curve defined by  $\vec{r}(t) = \langle 1, t^2, t^3 \rangle$ ,  $0 \leq t \leq 2\pi$ . Find the length of  $C$ .